

SIMPLICIAL COMPLEXES OBTAINED FROM QUALITATIVE PROBABILITY ORDERS

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ABSTRACT. In this paper we initiate the study of abstract simplicial complexes which are initial segments of qualitative probability orders. This is a natural class that contains the threshold complexes and is contained in the shifted complexes, but is equal to neither. In particular we construct a qualitative probability order on 26 atoms that has an initial segment which is not a threshold simplicial complex. Although 26 is probably not the minimal number for which such example exists we provide some evidence that it cannot be much smaller. We prove some necessary conditions for this class and make a conjecture as to a characterization of them. The conjectured characterization relies on some ideas from cooperative game theory.

1. INTRODUCTION

The concept of qualitative (comparative) probability takes its origins in attempts of de Finetti (1931) to axiomatise probability theory. It also played an important role in the expected utility theory of Savage (1954, p.32). The essence of a qualitative probability is that it does not give us numerical probabilities but instead provides us with the information, for every pair of events, which one is more likely to happen. The class of qualitative probability orders is broader than the class of probability measures for any $n \geq 5$ (Kraft et al., 1959). Qualitative probability orders on finite sets are now recognised as an important combinatorial object (Kraft et al., 1959; Fishburn, 1996, 1997) that finds applications in areas as far apart from probability theory as the theory of Gröbner bases (e.g., Maclagan, 1999).

Another important combinatorial object, also defined on a finite set is an abstract simplicial complex. This is a set of subsets of a finite set, called faces, with the property that a subset of a face is also a face. This concept is dual to the concept of a simple game whose winning coalitions form a set of subsets of a finite set with the property that if a coalition is winning, then every superset of it is also a winning coalition. The most studied class of simplicial complexes is the class of threshold simplicial complexes. These arise when we assign weights to elements of a finite set, set a threshold and define faces as those subsets whose combined weight is not achieving the threshold.

Given a qualitative probability order one may obtain a simplicial complex in an analogous way. For this one has to choose a threshold—which now will be a subset of our finite set—and consider as faces all subsets that are earlier than the threshold in the given qualitative probability order. This initial segment of the qualitative probability order will, in fact, be a simplicial complex. The collection of complexes arising as initial segments of probability orders contains threshold complexes and is contained in the well-studied class of shifted complexes (Klivans, 2005, 2007).

A natural question is therefore to ask if this is indeed a new class of complexes distinct from both the threshold complexes and the shifted ones.

In this paper we give an affirmative answer to both of these questions. We present an example of a shifted complex on 7 points that is not the initial segment of any qualitative probability order. On the other hand we also construct an initial segment of a qualitative probability order on 26 atoms that is not threshold. We also show that such example cannot be too small, in particular, it is unlikely that one can be found on fewer than 18 atoms.

The structure of this paper is as follows. In Section 2 we introduce the basics of qualitative probability orders. In Section 3 we consider abstract simplicial complexes and give necessary and sufficient conditions for them being threshold. In Section 4 we give a construction that will further provide us with examples of qualitative probability orders that are not related to any probability measure. Finally in Sections 5 and 6 we present our main result which is an example of a qualitative probability order on 26 atoms that is not threshold. Section 7 concludes with a conjectured characterization of initial segment complexes that is motivated by work in the theory of cooperative games.

2. QUALITATIVE PROBABILITY ORDERS AND DISCRETE CONES

In this paper all our objects are defined on the set $[n] = \{1, 2, \dots, n\}$. By $2^{[n]}$ we denote the set of all subsets of $[n]$. An order¹ \preceq on $2^{[n]}$ is called a *qualitative probability order* on $[n]$ if

$$(1) \quad \emptyset \preceq A$$

for every nonempty subset A of $[n]$, and \preceq satisfies de Finetti's axiom, namely for all $A, B, C \in 2^{[n]}$

$$(2) \quad A \preceq B \iff A \cup C \preceq B \cup C \text{ whenever } (A \cup B) \cap C = \emptyset.$$

Note that if we have a probability measure $\mathbf{p} = (p_1, \dots, p_n)$ on $[n]$, where p_i is the probability of i , then we know the probability $p(A)$ of every event A and $p(A) = \sum_{i \in A} p_i$. We may now define a relation \preceq on $2^{[n]}$ by

$$A \preceq B \text{ if and only if } p(A) \leq p(B);$$

obviously \preceq is a qualitative probability order on $[n]$, and any such order is called *representable* (e.g., Fishburn, 1996; Regoli, 2000). Those not obtainable in this way are called *non-representable*. The class of qualitative probability orders is broader than the class of probability measures for any $n \geq 5$ (Kraft et al., 1959). A non-representable qualitative probability order \preceq on $[n]$ is said to *almost agree* with the measure \mathbf{p} on $[n]$ if

$$(3) \quad A \preceq B \implies p(A) \leq p(B).$$

If such a measure \mathbf{p} exists, then the order \preceq is said to be *almost representable*. Since the arrow in (3) is only one-sided it is perfectly possible for an almost representable order to have $A \preceq B$ but not $B \preceq A$ while $p(A) = p(B)$.

We begin with some standard properties of qualitative probability orders which we will need subsequently. Let \preceq be a qualitative probability order on $2^{[n]}$. As

¹An order in this paper is any reflexive, complete and transitive binary relation. If it is also anti-symmetric, it is called linear order.

usual the following two relations can be derived from it. We write $A \prec B$ if $A \preceq B$ but not $B \preceq A$ and $A \sim B$ if $A \preceq B$ and $B \preceq A$.

Lemma 1. *Suppose that \preceq is a qualitative probability order on $2^{[n]}$, $A, B, C, D \in 2^{[n]}$, $A \preceq B$, $C \preceq D$ and $B \cap D = \emptyset$. Then $A \cup C \preceq B \cup D$. Moreover, if $A \prec B$ or $C \prec D$, then $A \cup C \prec B \cup D$.*

Proof. Firstly, let us consider the case when $A \cap C = \emptyset$. Let $B' = B \setminus C$ and $C' = C \setminus B$ and $I = B \cap C$. Then by (2) we have

$$A \cup C' \preceq B \cup C' = B' \cup C \preceq B' \cup D$$

where we have $A \cup C' \prec B' \cup D$ if $A \prec B$ or $C \prec D$. Now we have

$$A \cup C' \preceq B' \cup D \Leftrightarrow A \cup C = (A \cup C') \cup I \preceq (B' \cup D) \cup I = B \cup D.$$

Now let us consider the case when $A \cap C \neq \emptyset$. Let $A' = A \setminus C$. By (1) and (2) we now have $A' \prec B$. Since now we have $A' \cap C = \emptyset$ so by the previous case

$$A \cup C = A' \cup C \prec B \cup C \preceq B \cup D.$$

In this case we always obtain a strict inequality. \square

A weaker version of this lemma can be found in Maclagan (1999)[Lemma 2.2].

Definition 1. *A sequence of subsets $(A_1, \dots, A_j; B_1, \dots, B_j)$ of $[n]$ of even length $2j$ is said to be a trading transform of length j if for every $i \in [n]$*

$$|\{k \mid i \in A_k\}| = |\{k \mid i \in B_k\}|.$$

In other words, sets A_1, \dots, A_j can be converted into B_1, \dots, B_j by rearranging their elements. We say that an order \preceq on $2^{[n]}$ satisfies the k -th cancellation condition CC_k if there does not exist a trading transform $(A_1, \dots, A_k; B_1, \dots, B_k)$ such that $A_i \preceq B_i$ for all $i \in [k]$ and $A_i \prec B_i$ for at least one $i \in [k]$.

The key result of Kraft et al. (1959) can now be reformulated as follows.

Theorem 1 (Kraft-Pratt-Seidenberg). *A qualitative probability order \preceq is representable if and only if it satisfies CC_k for all $k = 1, 2, \dots$.*

It was also shown in Fishburn (1996, Section 2) that CC_2 and CC_3 hold for linear qualitative probability orders. It follows from de Finetti's axiom and properties of linear orders. It can be shown that a qualitative probability order satisfies CC_2 and CC_3 as well. Hence CC_4 is the first nontrivial cancellation condition. As was noticed in Kraft et al. (1959), for $n < 5$ all qualitative probability orders are representable, but for $n = 5$ there are non-representable ones. For $n = 5$ all orders are still almost representable Fishburn (1996) which is no longer true for $n = 6$ Kraft et al. (1959).

It will be useful for our constructions to rephrase some of these conditions in vector language. To every such linear order \preceq , there corresponds a *discrete cone* $C(\preceq)$ in T^n , where $T = \{-1, 0, 1\}$, as defined in Fishburn (1996).

Definition 2. *A subset $C \subseteq T^n$ is said to be a discrete cone if the following properties hold:*

- D1. $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \subseteq C$ and $\{-\mathbf{e}_1, \dots, -\mathbf{e}_n\} \cap C = \emptyset$, where $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis of \mathbb{R}^n ,
- D2. $\{-\mathbf{x}, \mathbf{x}\} \cap C \neq \emptyset$ for every $\mathbf{x} \in T^n$,

D3. $\mathbf{x} + \mathbf{y} \in C$ whenever $\mathbf{x}, \mathbf{y} \in C$ and $\mathbf{x} + \mathbf{y} \in T^n$.

We note that Fishburn (1996) requires $\mathbf{0} \notin C$ because his orders are anti-reflexive. In our case, condition D2 implies $\mathbf{0} \in C$.

Given a qualitative probability order \preceq on $2^{[n]}$, for every pair of subsets A, B satisfying $B \preceq A$ we construct a characteristic vector of this pair $\chi(A, B) = \chi(A) - \chi(B) \in T^n$. We define the set $C(\preceq)$ of all characteristic vectors $\chi(A, B)$, for $A, B \in 2^{[n]}$ such that $B \preceq A$. The two axioms of qualitative probability guarantee that $C(\preceq)$ is a discrete cone (see Fishburn, 1996, Lemma 2.1).

Following Fishburn (1996), the cancellation conditions can be reformulated as follows:

Proposition 1. *A qualitative probability order \preceq satisfies the k -th cancellation condition CC_k if and only if there does not exist a set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ of nonzero vectors in $C(\preceq)$ such that*

$$(4) \quad \mathbf{x}_1 + \mathbf{x}_2 + \dots + \mathbf{x}_k = \mathbf{0}$$

and $-\mathbf{x}_i \notin C(\preceq)$ for at least one i .

Geometrically, a qualitative probability order \preceq is representable if and only if there exists a positive vector $\mathbf{u} \in \mathbb{R}^n$ such that

$$\mathbf{x} \in C(\preceq) \iff (\mathbf{u}, \mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in T^n \setminus \{\mathbf{0}\},$$

where (\cdot, \cdot) is the standard inner product; that is, \preceq is representable if and only if every non-zero vector in the cone $C(\preceq)$ lies in the closed half-space $H_{\mathbf{u}}^+ = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{u}, \mathbf{x}) \geq 0\}$ of the corresponding hyperplane $H_{\mathbf{u}} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{u}, \mathbf{x}) = 0\}$.

Similarly, for a non-representable but *almost* representable qualitative probability order \preceq , there exists a vector $\mathbf{u} \in \mathbb{R}^n$ with non-negative entries such that

$$\mathbf{x} \in C(\preceq) \implies (\mathbf{u}, \mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in T^n \setminus \{\mathbf{0}\}.$$

In the latter case we can have $\mathbf{x} \in C(\preceq)$ and $-\mathbf{x} \notin C(\preceq)$ despite $(\mathbf{u}, \mathbf{x}) = 0$.

In both cases, the normalised vector \mathbf{u} gives us the probability measure, namely $\mathbf{p} = (u_1 + \dots + u_n)^{-1} (u_1, \dots, u_n)$, from which \preceq arises or with which it almost agrees.

3. SIMPLICIAL COMPLEXES AND THEIR CANCELLATION CONDITIONS

In this section we will introduce the objects of our study, simplicial complexes that arise as initial segments of a qualitative probability order. Using cancellation conditions for simplicial complexes, we will show that this class contains the threshold complexes and is contained in the shifted complexes. Using only these conditions it will be easy to show that the initial segment complexes are strictly contained in the shifted complexes. Showing the strict containment of the threshold complexes will require more elaborate constructions which will be developed in the rest of the paper.

A subset $\Delta \subseteq 2^{[n]}$ is an (*abstract*) *simplicial complex* if it satisfies the condition:

$$\text{if } B \in \Delta \text{ and } A \subseteq B, \text{ then } A \in \Delta.$$

Subsets that are in Δ are called *faces*. Abstract simplicial complexes arose from geometric simplicial complexes in topology (e.g., Maunder, 1996). Indeed, for every geometric simplicial complex Δ the set of vertex sets of simplices in Δ is an abstract simplicial complex, also called the *vertex scheme* of Δ . In combinatorial

optimization various abstract simplicial complexes associated with finite graphs (Jonsson (2005)) are studied, such as the independence complex, matching complex etc. Abstract simplicial complexes are also in one-to-one correspondence with *simple games* as defined by Neumann & Morgenstern (1944). A simple game is a pair $G = ([n], W)$, where W is a subset of the power set $2^{[n]}$ which satisfies the monotonicity condition:

$$\text{if } X \in W \text{ and } X \subseteq Y \subseteq [n], \text{ then } Y \in W.$$

The subsets from W are called *winning coalitions* and the subsets from $L = 2^{[n]} \setminus W$ are called *losing coalitions*. Obviously the set of losing coalitions L is a simplicial complex. The reverse is also true: if Δ is a simplicial complex, then the set $2^{[n]} \setminus \Delta$ is a set of winning coalitions of a certain simple game.

A well-studied class of simplicial complexes is the *threshold* complexes (mostly as an equivalent concept to the concept of a weighted majority game but also as threshold hypergraphs (Reiterman et al., 1985)). A simplicial complex Δ is a threshold complex if there exist non-negative reals w_1, \dots, w_n and a positive constant q , such that

$$A \in \Delta \iff w(A) = \sum_{i \in A} w_i < q.$$

The same parameters define a *weighted majority game* by setting

$$A \in W \iff w(A) = \sum_{i \in A} w_i \geq q.$$

This game has the standard notation $[q; w_1, \dots, w_n]$.

A much larger but still well-understood class of simplicial complexes are *shifted* simplicial complexes (Klivans, 2005, 2007). A simplicial complex is shifted if there exists an order \preceq on the set of vertices $[n]$ such that for any face F , replacing any of its vertices $x \in F$ with a vertex y such that $y \preceq x$ results in a subset $(F \setminus \{x\}) \cup \{y\}$ which is also a face. Shifted complexes correspond to complete² games (Freixas & Molinero, 2009). A complete game has an order \preceq on players such that if a coalition W is winning, then replacing any player $x \in W$ with a player $x \preceq z$ results in a coalition $(W \setminus \{x\}) \cup \{z\}$ which is also winning.

A related concept is the so-called Isbel's desirability relation \leq_I Taylor & Zwicker (1999). Given a game G the relation \leq_I on $[n]$ is defined by setting $j \leq_I i$ if for every set $X \subseteq [n]$ not containing i and j

$$(5) \quad X \cup \{j\} \in W \implies X \cup \{i\} \in W.$$

The idea is that if $j \leq_I i$, then i is more desirable as a coalition partner than j . The game is complete if and only if \leq_I is an order on $[n]$.

Let \preceq be a qualitative probability order on $[n]$ and $T \in 2^{[n]}$. We denote

$$\Delta(\preceq, T) = \{X \subseteq [n] \mid X \prec T\},$$

where $X \prec Y$ stands for $X \preceq Y$ but not $Y \preceq X$, and call it an *initial segment* of \preceq .

Lemma 2. *Any initial segment of a qualitative probability order is a simplicial complex.*

²sometimes also called linear

Proof. Suppose that $\Delta = \Delta(\preceq, T)$ and $B \in \Delta$. If $A \subset B$, then let $C = B \setminus A$. By (1) we have that $\emptyset \preceq C$ and since $A \cap C = \emptyset$ it follows from (2) that $\emptyset \cup A \preceq C \cup A$ which implies that $A \preceq B$. Since Δ is an initial segment, $B \in \Delta$ and $A \preceq B$ implies that $A \in \Delta$ and thus Δ is a simplicial complex. \square

We will refer to simplicial complexes that arise as initial segments of some qualitative probability order as an *initial segment complex*.

In a similar manner as for the qualitative probability orders, cancellation conditions will play a key role in our analyzing simplicial complexes.

Definition 3. A simplicial complex Δ is said to satisfy CC_k^* if for no $k \geq 2$ there exists a trading transform $(A_1, \dots, A_k; B_1, \dots, B_k)$, such that $A_i \in \Delta$ and $B_i \notin \Delta$, for every $i \in [k]$.

Let us show the connection between CC_k and CC_k^* .

Theorem 2. Suppose \preceq is a qualitative probability order on $2^{[n]}$ and $\Delta(\preceq, T)$ is its initial segment. If \preceq satisfies CC_k then $\Delta(\preceq, T)$ satisfies CC_k^* .

This gives us some initial properties of initial segment complexes. Since conditions CC_k , $k = 2, 3$, hold for all qualitative probability orders (Fishburn, 1996) we obtain

Theorem 3. If an abstract simplicial complex $\Delta \subseteq 2^{[n]}$ is an initial segment complex, then it satisfies CC_k^* for all $k \leq 3$.

From this theorem we get the following corollary, due to Caroline Klivans (personal communication):

Corollary 1. Every initial segment complex is a shifted complex. Moreover, there are shifted complexes that are not initial segment complexes.

Proof. Let Δ be a non-shifted simplicial complex. then it is known to contain an obstruction of the form: there are $i, j \in [n]$, and $A, B \in \Delta$, neither containing i or j , so that $A \cup i$ and $B \cup j$ are in Δ but neither $i \cup B$ nor $j \cup A$ are in Δ (Klivans, 2005). But then $(A \cup i, B \cup j; B \cup i, A \cup j)$ is a trading transform that violates CC_2^* . Since all initial segments satisfy CC_2^* they must all be shifted.

On the other hand, there are shifted complexes that fail to satisfy CC_2^* and hence can not be initial segments. Let Δ be the smallest shifted complex (where shifting is with respect to the usual ordering) that contains $\{1, 5, 7\}$ and $\{2, 3, 4, 6\}$. Then it is easy to check that neither $\{3, 4, 7\}$ nor $\{1, 2, 5, 6\}$ are in Δ but

$$(6) \quad (\{1, 5, 7\}, \{2, 3, 4, 6\}; \{3, 4, 7\}, \{1, 2, 5, 6\})$$

is a transform in violation of CC_2^* . \square

Similarly, the *terminal segment*

$$G(\preceq, T) = \{X \subseteq [n] \mid T \preceq X\}$$

of any qualitative probability order is a complete simple game.

The Theorem 2.4.2 of the book Taylor & Zwicker (1999) can be reformulated to give necessary and sufficient conditions for the simplicial complex to be a threshold.

Theorem 4. An abstract simplicial complex $\Delta \subseteq 2^{[n]}$ is a threshold complex if and only if the condition CC_k^* holds for all $k \geq 2$.

Above we showed that the initial segment complexes are strictly contained in the shifted complexes. What is the relationship between the initial segment complexes and threshold complexes?

Lemma 3. *Every threshold complex is an initial segment complex.*

Proof. The threshold complex defined by the weights w_1, \dots, w_n and a positive constant q is the initial segment of the representable qualitative probability order whose where $p_i = w_i$, $1 \leq i \leq n$ and where the threshold set T has the property that $w(A) \leq w(T) < q$ for all $A \in \Delta$. \square

This leaves us with the question of whether this containment is strict, i.e., are there initial segment complexes which are not threshold complexes. One might think that some initial segment of a non-representable qualitative probability order is not threshold. Unfortunately that may not be the case.

Example 1. *This example, adapted from (MacLagan, 1999)[Example 2.5, Example 3.9] gives a non-representable qualitative probability order for which every initial segment complex is threshold. Construct a representable qualitative probability order on $2^{[5]}$ using the p_i 's $\{7, 10, 16, 20, 22\}$. The order begins*

$$(7) \quad \emptyset \prec 1 \prec 2 \prec 3 \prec 12 \prec 4 \prec 5 \prec \dots$$

where 1 denotes the singleton set $\{1\}$ and by 12 we mean $\{1, 2\}$. Since the qualitative probability order is representable, every initial segment is a threshold complex. Now suppose we interchange the order of 12 and 4. The new ordering, which begins

$$(8) \quad \emptyset \prec 1 \prec 2 \prec 3 \prec 4 \prec 12 \prec 5 \prec \dots,$$

is still a qualitative probability order but it is no longer representable (MacLagan, 1999, Example 2.5). With one exception, all of the initial segments in this new non-representable qualitative order are initial segments in the original one and thus are threshold. The one exception is the segment

$$(9) \quad \emptyset \prec 1 \prec 2 \prec 3 \prec 4$$

which is obviously a threshold complex.

Another approach to finding an initial segment complex that is not threshold is to construct a complex that violates CC_k^* for some small value of k . As noted above, all initial segment complexes satisfy CC_2^* and CC_3^* so the smallest condition that could fail is CC_4^* . We will now show that for small values of n cancellation condition CC_4^* is satisfied for any initial segment. This will also give us invaluable information on how to construct a non-threshold initial segment later.

Definition 4. *Two pairs of subsets (A_1, B_1) and (A_2, B_2) are said to be compatible if the following two conditions hold:*

$$\begin{aligned} x \in A_1 \cap A_2 &\implies x \in B_1 \cup B_2, \text{ and} \\ x \in B_1 \cap B_2 &\implies x \in A_1 \cup A_2. \end{aligned}$$

Lemma 4. *Let \preceq be a qualitative probability order on $2^{[n]}$, $T \subseteq [n]$, and let $\Delta = \Delta_n(\preceq, T)$ be the respective initial segment. Suppose $(A_1, \dots, A_s, B_1, \dots, B_s)$ is a trading transform and $A_i \prec T \preceq B_j$ for all $i, j \in [s]$. If any two pairs (A_i, B_k) and (A_j, B_l) are compatible, then \preceq fails to satisfy CC_{s-1} .*

Proof. Let us define

$$(10) \quad \bar{A}_i = A_i \setminus (A_i \cap B_k), \quad \bar{B}_k = B_k \setminus (A_i \cap B_k),$$

$$(11) \quad \bar{A}_j = A_j \setminus (A_j \cap B_l), \quad \bar{B}_l = B_l \setminus (A_j \cap B_l).$$

We note that

$$(12) \quad \bar{A}_i \cap \bar{A}_j = \bar{B}_k \cap \bar{B}_l = \emptyset.$$

Indeed, suppose, for example, $x \in \bar{A}_i \cap \bar{A}_j$, then also $x \in A_i \cap A_j$ and by the compatibility $x \in B_k$ or $x \in B_l$. In both cases it is impossible for x to be in $x \in \bar{A}_i \cap \bar{A}_j$. We note also that by Lemma 1 we have

$$(13) \quad \bar{A}_i \cup \bar{A}_j \prec \bar{B}_k \cup \bar{B}_l.$$

Now we observe that

$$(\bar{A}_i, \bar{A}_j, A_{m_1}, \dots, A_{m_{s-2}}; \bar{B}_k, \bar{B}_l, B_{r_1}, \dots, B_{r_{s-2}}).$$

is a trading transform. Hence, due to (12),

$$(\bar{A}_i \cup \bar{A}_j, A_{m_1}, \dots, A_{m_{s-2}}; \bar{B}_k \cup \bar{B}_l, B_{r_1}, \dots, B_{r_{s-2}})$$

is also a trading transform. This violates CC_{s-1} since (13) holds and $A_{m_t} \prec B_{r_t}$ for all $t = 1, \dots, s-2$. \square

By definition of a trading transform we are allowed to use repetitions of the same coalition in it. However we will show that to violate CC_4^* we need a trading transform $(A_1, \dots, A_4; B_1, \dots, B_4)$ where all A 's and B 's are different.

Lemma 5. *Let \preceq be a qualitative probability order on $2^{[n]}$, $T \subseteq [n]$, and let $\Delta = \Delta_n(\preceq, T)$ be the respective initial segment. Suppose $(A_1, \dots, A_4, B_1, \dots, B_4)$ is a trading transform and $A_i \prec T \preceq B_j$ for all $i, j \in [4]$. Then*

$$|\{A_1, \dots, A_4\}| = |\{B_1, \dots, B_4\}| = 4.$$

Proof. Note that every pair $(A_i, B_j), (A_l, B_k)$ is not compatible. Otherwise by Lemma 4 the order \preceq fails CC_3 , which contradicts to the fact that every qualitative probability satisfies CC_3 . Assume, to the contrary, that we have at least two identical coalitions among A_1, \dots, A_4 or B_1, \dots, B_4 . Without loss of generality we can assume $A_1 = A_2$. Clearly all A 's or all B 's cannot coincide and there are at least two different A 's and two different B 's. Suppose $A_1 \neq A_3$ and $B_1 \neq B_2$. The pair $(A_1, B_1), (A_3, B_2)$ is not compatible. It means one of the following two statements is true: either there is $x \in A_1 \cap A_3$ such that $x \notin B_1 \cup B_2$ or there is $y \in B_1 \cap B_2$ such that $y \notin A_1 \cup A_3$. Consider the first case the other one is similar. We know that $x \in A_1 \cap A_3$ and we have at least three copies of x among A_1, \dots, A_4 . At the same time $x \notin B_1 \cup B_2$ and there could be at most two copies of x among B_1, \dots, B_4 . This is a contradiction. \square

Theorem 5. CC_4^* holds for $\Delta = \Delta_n(\preceq, T)$ for all $n \leq 17$.

Proof. Let us consider the set of column vectors

$$(14) \quad U = \{\mathbf{x} \in \mathbb{R}^8 \mid x_i \in \{0, 1\} \text{ and } x_1 + x_2 + x_3 + x_4 = x_5 + x_6 + x_7 + x_8 = 2\}.$$

This set has an involution $\mathbf{x} \mapsto \bar{\mathbf{x}}$, where $\bar{x}_i = 1 - x_i$. Say, if $\mathbf{x} = (1, 1, 0, 0, 0, 0, 1, 1)^T$, then $\bar{\mathbf{x}} = (0, 0, 1, 1, 1, 1, 0, 0)^T$. There are 36 vectors from U which are split into 18 pairs $\{\mathbf{x}, \bar{\mathbf{x}}\}$.

Suppose now $\mathcal{T} = (A_1, A_2, A_3, A_4; B_1, B_2, B_3, B_4)$ is a trading transform, $A_i \prec T \preceq B_j$ and no two coalitions in the trading transform coincide. Let us write the characteristic vectors of $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$ as rows of $8 \times n$ matrix M , respectively. Since \preceq satisfies CC_3 , by Lemma 4 we know that no two pairs (A_i, B_a) and (A_j, B_b) are compatible. The same can be said about the complementary pair of pairs (A_k, B_c) and (A_l, B_d) , where $\{a, b, c, d\} = \{i, j, h, l\} = [4]$. We have

$$A_i \prec B_a, A_j \prec B_b, A_h \prec B_c, A_l \prec B_d,$$

Since (A_i, B_a) and (A_j, B_b) are not compatible one of the following two statements is true: either there exists $x \in A_i \cap A_j$ such that $x \notin B_a \cup B_b$ or there exists $y \in B_a \cap B_b$ such that $x \notin A_i \cup A_j$. As \mathcal{T} is the trading transform in the first case we will also have $x \in B_c \cap B_d$ such that $x \notin A_h \cup A_l$; in the second $y \in A_h \cap A_l$ such that $y \notin B_c \cup B_d$.

Let us consider two columns M_x and M_y of M that corresponds to elements $x, y \in [n]$. The above considerations show that both belong to U and $M_x = \bar{M}_y$.

In particular, if $(i, j, k, l) = (a, b, c, d) = (1, 2, 3, 4)$, then the columns M_x and M_y will be as in the following picture

$$M = \begin{bmatrix} \chi(A_1) \\ \chi(A_2) \\ \chi(A_3) \\ \chi(A_4) \\ \chi(B_1) \\ \chi(B_2) \\ \chi(B_3) \\ \chi(B_4) \end{bmatrix} = \begin{array}{c|c} \begin{matrix} x & y \end{matrix} \\ \begin{matrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ \hline 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{matrix} \end{array}$$

(we emphasize however that we have only one such column in the matrix, not both). We saw that one pairing of indices $(i, a), (j, b), (k, c), (l, d)$ gives us a column from one of the 18 pairs of U . It is easy to see that a vector from every pair of U can be obtained by the appropriate choice of the pairing of indices. This means that the matrix contains at least 18 columns. That is $n \geq 18$. \square

While no initial segment complex on fewer than 18 points can fail CC_4^* , there is such an example on 26 points which will show that the initial segment complexes strictly contain the threshold complexes. The next three sections are devoted to constructing such an example. The next section presents a general construction technique for producing almost representable qualitative probability orders from representable ones. This technique will be employed in section 5 to construct our example. Some of the proofs required will be done in section 6.

4. CONSTRUCTING ALMOST REPRESENTABLE ORDERS FROM NONLINEAR REPRESENTABLE ONES

Our approach to finding an initial segment complex that is not threshold will be to start with a non-linear representable qualitative probability order and then perturb it so as to produce an almost representable order. By judicious breaking of ties in this new order we will be able to produce an initial segment that will violate CC_4^* . The language of discrete cones will be helpful and we begin with a technical lemma that will be needed in the construction.

Proposition 2. *Let \preceq be a non-representable but almost representable qualitative probability order which almost agrees with a probability measure \mathbf{p} . Suppose that the m th cancellation condition CC_m is violated, and that for some non-zero vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\} \subseteq C(\preceq)$ the condition (4) holds, i.e., $\mathbf{x}_1 + \dots + \mathbf{x}_m = \mathbf{0}$ and $\mathbf{x}_i \notin C(\preceq)$ for at least one $i \in [m]$. Then all of the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ lie in the hyperplane $H_{\mathbf{p}}$.*

Proof. First note that for every $\mathbf{x} \in C(\preceq)$ which does not belong to $H_{\mathbf{p}}$, we have $(\mathbf{p}, \mathbf{x}) > 0$. Hence the condition (4) can hold only when all $\mathbf{x}_i \in H_{\mathbf{p}}$. \square

We need to understand how we can construct new qualitative probability orders from old ones so we need the following investigation. Let \preceq be a representable but not linear qualitative probability order which agrees with a probability measure \mathbf{p} .

Let $S(\preceq)$ be the set of all vectors of $C(\preceq)$ which lie in the corresponding hyperplane $H_{\mathbf{p}}$. Clearly, if $\mathbf{x} \in S(\preceq)$, then $-\mathbf{x}$ is a vector of $S(\preceq)$ as well. Since in the definition of discrete cone it is sufficient that only one of these vectors is in $C(\preceq)$ we may try to remove one of them in order to obtain a new qualitative probability order. The new order will almost agree with \mathbf{p} and hence will be at least almost representable. The big question is: what are the conditions under which a set of vectors can be removed from $S(\preceq)$?

What can prevent us from removing a vector from $S(\preceq)$? Intuitively, we cannot remove a vector if the set comparison corresponding to it is a consequence of those remaining. We need to consider what a consequence means formally.

There are two ways in which one set comparison might imply another one. The first way is by means of the de Finetti condition. This however is already built in the definition of the discrete cone as $\chi(A, B) = \chi(A \cup C, B \cup C)$. Another way in which a comparison may be implied from two other is transitivity. This has a nice algebraic characterisation. Indeed, if $C \prec B \prec A$, then $\chi(A, C) = \chi(A, B) + \chi(B, C)$. This leads us to the following definition.

Following Christian et al. (2007) let us define a restricted sum for vectors in a discrete cone C . Let $\mathbf{u}, \mathbf{v} \in C$. Then

$$\mathbf{u} \oplus \mathbf{v} = \begin{cases} \mathbf{u} + \mathbf{v} & \text{if } \mathbf{u} + \mathbf{v} \in T^n, \\ \text{undefined} & \text{if } \mathbf{u} + \mathbf{v} \notin T^n. \end{cases}$$

It was shown in (Fishburn, 1996, Lemma 2.1) that the transitivity of a qualitative probability order is equivalent to closedness of its corresponding discrete cone with respect to the restricted addition (without formally defining the latter). The axiom D3 of the discrete cone can be rewritten as

D3. $\mathbf{x} \oplus \mathbf{y} \in C$ whenever $\mathbf{x}, \mathbf{y} \in C$ and $\mathbf{x} \oplus \mathbf{y}$ is defined.

Note that a restricted sum is not associative.

Theorem 6 (Construction method). *Let \preceq be a representable non-linear qualitative probability order which agrees with the probability measure \mathbf{p} . Let $S(\preceq)$ be the set of all vectors of $C(\preceq)$ which lie in the hyperplane $H_{\mathbf{p}}$. Let X be a subset of $S(\preceq)$ such that*

- $X \cap \{\mathbf{s}, -\mathbf{s}\} \neq \emptyset$ for every $\mathbf{s} \in S(\preceq)$.
- X is closed under the operation of restricted sum.

Then $Y = S(\preceq) \setminus X$ may be dropped from $C(\preceq)$, that is $C_Y = C(\preceq) \setminus Y$ is a discrete cone.

Proof. We first note that if $\mathbf{x} \in C(\preceq) \setminus S(\preceq)$ and $\mathbf{y} \in C(\preceq)$, then $\mathbf{x} \oplus \mathbf{y}$, if defined, cannot be in $S(\preceq)$. So due to closedness of X under the restricted addition all axioms of a discrete cone are satisfied for C_Y . On the other hand, if for some two vectors $\mathbf{x}, \mathbf{y} \in X$ we have $\mathbf{x} \oplus \mathbf{y} \in Y$, then C_Y would not be a discrete cone and we would not be able to construct a qualitative probability order associated with this set. \square

Example 2 (Positive example). *The probability measure*

$$\mathbf{p} = \frac{1}{16}(6, 4, 3, 2, 1).$$

defines a qualitative probability order \preceq on [5] (which is better written from the other end):

$$\emptyset \prec 5 \prec 4 \prec 3 \prec 45 \prec 35 \sim 2 \prec 25 \sim 34 \prec 1 \prec 345 \sim 24 \prec 23 \sim 15 \prec 245 \prec 14 \sim 235 \dots$$

(Here only the first 17 terms are shown, since the remaining ones can be uniquely reconstructed. See (Kraft et al., 1959, Proposition 1) for details). There are only four equivalences here

$$35 \sim 2, \quad 25 \sim 34, \quad 23 \sim 15 \quad \text{and} \quad 14 \sim 235,$$

and all other follow from them, that is:

$$35 \sim 2 \text{ implies } 345 \sim 24, \quad 135 \sim 12;$$

$$25 \sim 34 \text{ implies } 125 \sim 134;$$

$$23 \sim 15 \text{ implies } 234 \sim 145;$$

$$14 \sim 235 \text{ has no consequences}$$

Let $\mathbf{u}_1 = \chi(2, 35) = (0, 1, -1, 0, -1)$, $\mathbf{u}_2 = \chi(34, 25) = (0, -1, 1, 1, -1)$, $\mathbf{u}_3 = \chi(15, 23) = (1, -1, -1, 0, 1)$ and $\mathbf{u}_4 = \chi(235, 14) = (-1, 1, 1, -1, 1)$. Then

$$S(\preceq) = \{\pm \mathbf{u}_1, \pm \mathbf{u}_2, \pm \mathbf{u}_3, \pm \mathbf{u}_4\}$$

and $X = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is closed under the restricted addition as $\mathbf{u}_i \oplus \mathbf{u}_j$ is undefined for all $i \neq j$. Note that $\mathbf{u}_i \oplus -\mathbf{u}_j$ is also undefined for all $i \neq j$. Hence we can subtract from the cone $C(\preceq)$ any non-empty subset Y of $-X = \{-\mathbf{u}_1, -\mathbf{u}_2, -\mathbf{u}_3, -\mathbf{u}_4\}$ and still get a qualitative probability. Since

$$\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}.$$

it will not be representable. The new order corresponding to the discrete cone C_{-X} is linear.

Example 3 (Negative example). *A certain qualitative probability order is associated with the Gabelman game of order 3. Nine players are involved each of whom we think as associated with a certain cell of a 3×3 square:*

1	2	3
4	5	6
7	8	9

The i th player is given a positive weight w_i , $i = 1, 2, \dots, 9$, such that in the qualitative probability order, associated with $\mathbf{w} = (w_1, \dots, w_9)$,

$$147 \sim 258 \sim 369 \sim 123 \sim 456 \sim 789.$$

Suppose that we want to construct a qualitative probability order \preceq for which

$$147 \sim 258 \sim 369 \prec 123 \sim 456 \sim 789.$$

Then we would like to claim that it is not weighted since for the vectors

$$\mathbf{x}_1 = (0, 1, 1, -1, 0, 0, -1, 0, 0) = \chi(123, 147),$$

$$\mathbf{x}_2 = (0, -1, 0, 1, 0, 1, 0, -1, 0) = \chi(456, 258),$$

$$\mathbf{x}_3 = (0, 0, -1, 0, 0, -1, 1, 1, 0) = \chi(789, 369)$$

we have $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = \mathbf{0}$. Putting the sign \prec instead of \sim between 369 and 123 will also automatically imply $147 \prec 123$, $258 \prec 456$ and $369 \prec 789$. This means that we are dropping the set of vectors $\{-\mathbf{x}_1, -\mathbf{x}_2, -\mathbf{x}_3\}$ from the cone while leaving the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ there. This would not be possible since $\mathbf{x}_1 \oplus \mathbf{x}_2 = -\mathbf{x}_3$. So every $X \supset \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $X \cap \{-\mathbf{x}_1, -\mathbf{x}_2, -\mathbf{x}_3\} = \emptyset$ is not closed under \oplus .

5. AN EXAMPLE OF A NONTHRESHOLD INITIAL SEGMENT OF A LINEAR QUALITATIVE PROBABILITY ORDER

In this section we shall construct an almost representable linear qualitative probability order \sqsubseteq on $2^{[26]}$ and a subset $T \subseteq [26]$, such that the initial segment $\Delta(\sqsubseteq, T)$ of \sqsubseteq is not a threshold complex as it fails to satisfy the condition CC_4^* .

The idea of the example is as follows. We will start with a representable linear qualitative probability order \preceq on $[18]$ defined by weights w_1, \dots, w_{18} and extend it to a representable but nonlinear qualitative probability order \preceq' on $[26]$ with weights w_1, \dots, w_{26} . A distinctive feature of \preceq' will be the existence of eight sets $A'_1, \dots, A'_4, B'_1, \dots, B'_4$ in $[26]$ such that:

- (1) The sequence $(A'_1, \dots, A'_4; B'_1, \dots, B'_4)$ is a trading transform.
- (2) The sets $A'_1, \dots, A'_4, B'_1, \dots, B'_4$ are tied in \preceq' , that is,

$$A'_1 \sim' \dots A'_4 \sim' B'_1 \sim' \dots B'_4.$$

- (3) If any two distinct sets $X, Y \subseteq [26]$ are tied in \preceq' , then $\chi(X, Y) = \chi(S, T)$, where $S, T \in \{A'_1, \dots, A'_4, B'_1, \dots, B'_4\}$. In other words all equivalences in \preceq' are consequences of $A'_i \sim' A'_j$, $A'_i \sim' B'_j$, $B'_i \sim' B'_j$, where $i, j \in [4]$.

Then we will use Theorem 6 to untie the eight sets and to construct a comparative probability order \sqsubseteq for which

$$A'_1 \sqsubset A'_2 \sqsubset A'_3 \sqsubset A'_4 \sqsubset B'_1 \sqsubset B'_2 \sqsubset B'_3 \sqsubset B'_4,$$

where $X \sqsubset Y$ means that $X \sqsubseteq Y$ is true but not $Y \sqsubseteq X$.

This will give us an initial segment $\Delta(\sqsubseteq, B'_1)$ of the linear qualitative probability order \sqsubseteq , which is not threshold since CC_4^* fails to hold.

Let \preceq be a representable linear qualitative probability order on $2^{[18]}$ with weights w_1, \dots, w_{18} that are linearly independent (over \mathbb{Z}) real numbers in the interval $[0, 1]$. Due to the choice of weights, no two distinct subsets $X, Y \subseteq [18]$ have equal weights relative to this system of weights, i.e.,

$$X \neq Y \implies w(X) = \sum_{i \in X} w_i \neq w(Y) = \sum_{i \in Y} w_i.$$

Let us consider again the set U defined in (14). Let M be a subset of U with the following properties: $|M| = 18$ and $\mathbf{x} \in M$ if and only if $\bar{\mathbf{x}} \notin M$. In other words M contains exactly one vector from every pair into which U is split. By

M we will also denote an 8×18 matrix whose columns are all the vectors from M taken in arbitrary order. By $A_1, \dots, A_4, B_1, \dots, B_4$ we denote the sets with characteristic vectors equal to the rows M_1, \dots, M_8 of M , respectively. The way M was constructed secures that the following lemma is true.

Lemma 6. *The subsets $A_1, \dots, A_4, B_1, \dots, B_4$ s of [18] satisfy:*

- (1) $(A_1, \dots, A_4; B_1, \dots, B_4)$ is a trading transform;
- (2) for any choice of $i, k, j, m \in [4]$ with $i \neq k$ and $j \neq m$ the pair $(A_i, B_j), (A_k, B_m)$ is not compatible.

We shall now embed $A_1, \dots, A_4, B_1, \dots, B_4$ into [26] and add new elements to them forming $A'_1, \dots, A'_4, B'_1, \dots, B'_4$ in such a way that the characteristic vectors $\chi(A'_1), \dots, \chi(A'_4), \chi(B'_1), \dots, \chi(B'_4)$ are the rows M'_1, \dots, M'_8 of the following matrix

$$(15) \quad M' = \begin{bmatrix} & \begin{matrix} 1 \dots 18 \end{matrix} & \begin{matrix} 19 & 20 & 21 & 22 \end{matrix} & \begin{matrix} 23 & 24 & 25 & 26 \end{matrix} \\ \begin{matrix} \chi(A_1) \\ \chi(A_2) \\ \chi(A_3) \\ \chi(A_4) \end{matrix} & & \begin{matrix} I \\ \\ \\ \end{matrix} & \begin{matrix} \\ \\ \\ I \end{matrix} \\ \hline \begin{matrix} \chi(B_1) \\ \chi(B_2) \\ \chi(B_3) \\ \chi(B_4) \end{matrix} & & \begin{matrix} \\ \\ J \\ \end{matrix} & \begin{matrix} \\ \\ \\ I \end{matrix} \end{bmatrix},$$

respectively. Here I is the 4×4 identity matrix and

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that if X belongs to [18], it also belongs to [26], so the notation $\chi(X)$ is ambiguous as it may be a vector from \mathbb{Z}^{18} or from \mathbb{Z}^{26} , depending on the circumstances. However the reference set will be always clear from the context and the use of this notation will create no confusion.

One can see that $(A'_1, \dots, A'_4; B'_1, \dots, B'_4)$ is again a trading transform and there are no compatible pairs (A'_i, B'_j) , (A'_k, B'_m) , where $i, k, j, m \in [4]$ and $i \neq k$ or $j \neq m$. We shall now choose weights w_{19}, \dots, w_{26} of new elements $19, \dots, 26$ in such a way that the sets $A'_1, A'_2, A'_3, A'_4, B'_1, B'_2, B'_3, B'_4$ all have the same weight N , which is a sufficiently large number. It will be clear from the proof how large it should be.

To find weights w_{19}, \dots, w_{26} that satisfy this condition we need to solve the following system of linear equations

$$(16) \quad \begin{pmatrix} I & I \\ J & I \end{pmatrix} \begin{pmatrix} w_{19} \\ \vdots \\ w_{26} \end{pmatrix} = N\mathbf{1} - M \cdot \mathbf{w},$$

where $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^8$ and $\mathbf{w} = (w_1, \dots, w_{18})^T \in \mathbb{R}^{18}$.

The matrix from (16) has rank 7, and the augmented matrix of the system has the same rank. Therefore, the solution set is not empty, moreover, there is one free variable (and any one can be chosen for this role). Let this free variable be w_{26} and let us give it value K , such that K is large but much smaller than N . In

particular, $126 < K < N$. Now we can express all other weights w_{19}, \dots, w_{25} in terms of $w_{26} = K$ as follows:

$$\begin{aligned}
 w_{19} &= N - K - (\chi(A_4) - \chi(B_1) + \chi(A_1)) \cdot \mathbf{w} \\
 w_{20} &= N - K - (\chi(A_4) - \chi(B_1) + \chi(A_1) - \chi(B_2) + \chi(A_2)) \cdot \mathbf{w} \\
 w_{21} &= N - K - (\chi(A_4) - \chi(B_1) + \chi(A_1) - \chi(B_2) + \chi(A_2) - \\
 &\quad \chi(B_3) + \chi(A_3)) \cdot \mathbf{w} \\
 w_{22} &= N - K - \chi(A_4) \cdot \mathbf{w} \\
 w_{23} &= K - (-\chi(A_4) + \chi(B_1)) \cdot \mathbf{w} \\
 w_{24} &= K - (-\chi(A_4) + \chi(B_1) - \chi(A_1) + \chi(B_2)) \cdot \mathbf{w} \\
 w_{25} &= K - (-\chi(A_4) + \chi(B_1) - \chi(A_1) + \chi(B_2) - \chi(A_2) + \chi(B_3)) \cdot \mathbf{w}.
 \end{aligned}
 \tag{17}$$

By choice of N and K weights w_{19}, \dots, w_{25} are positive. Indeed, all “small” terms in the right-hand-side of (17) are strictly less than $7 \cdot 18 = 126 < \min\{K, N - K\}$.

Let \preceq' be the representable qualitative probability order on $[26]$ defined by the weight vector $\mathbf{w}' = (w_1, \dots, w_{26})$. Using \preceq' we would like to construct a linear qualitative probability order \sqsubseteq on $2^{[26]}$ that ranks the subsets A'_i and B'_j in the sequence

$$A'_1 \sqsubset A'_2 \sqsubset A'_3 \sqsubset A'_4 \sqsubset B'_1 \sqsubset B'_2 \sqsubset B'_3 \sqsubset B'_4.
 \tag{18}$$

We will make use of Theorem 6 now. Let $H_{\mathbf{w}'} = \{x \in \mathbb{R}^n \mid (\mathbf{w}', x) = 0\}$ be the hyperplane with the normal vector \mathbf{w}' and $S(\preceq')$ be the set of all vectors of the respective discrete cone $C(\preceq')$ that lie in $H_{\mathbf{w}'}$. Suppose

$$X' = \{\chi(C, D) \mid C, D \in \{A'_1, \dots, A'_4, B'_1, \dots, B'_4\} \text{ and } D \text{ earlier than } C \text{ in (18)}\}.$$

This is a subset of T^{26} , where $T = \{-1, 0, 1\}$. Let also $Y' = S(\preceq') \setminus X'$. To use Theorem 6 with the goal to achieve (18) we need to show, that

- $S(\preceq') = X' \cup -X'$ and
- X' is closed under the operation of restricted sum.

If we could prove this, then $C(\sqsubseteq) = C(\preceq') \setminus Y'$ is a discrete cone of a linear qualitative probability order \sqsubseteq on $[26]$ satisfying (18). Then the initial segment $\Delta(\sqsubseteq, B'_1)$ will not be a threshold complex, because the condition CC_4^* will fail for it.

Let Y be one of the sets $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$. By \check{Y} we will denote the corresponding superset of Y from the set $\{A'_1, A'_2, A'_3, A'_4, B'_1, B'_2, B'_3, B'_4\}$.

Proposition 3. *The subset*

$$X = \{\chi(C, D) \mid C, D \in \{A_1, \dots, A_4, B_1, \dots, B_4\} \text{ with } \check{D} \text{ earlier than } \check{C} \text{ in (18)}\}.$$

of T^{18} is closed under the operation of restricted sum.

Proof. Let \mathbf{u} and \mathbf{v} be any two vectors in X . As we will see the restricted sum $\mathbf{u} \oplus \mathbf{v}$ is almost always undefined. Without loss of generality we can consider only five cases.

Case 1. $\mathbf{u} = \chi(B_i, A_j)$ and $\mathbf{v} = \chi(B_k, A_m)$, where $i \neq k$ and $j \neq m$. In this case by Lemma 6 the pairs (B_i, A_j) and (B_k, A_m) are not compatible. It means that there exists $p \in [18]$ such that either $p \in B_i \cap B_k$ and $p \notin A_j \cup A_m$ or $p \in A_j \cap A_m$

and $p \notin B_i \cup B_k$. The vector $\mathbf{u} + \mathbf{v}$ has 2 or -2 at p th position and $\mathbf{u} \oplus \mathbf{v}$ is undefined. This is illustrated in the table below:

	$\chi(B_i)$	$\chi(B_k)$	$\chi(A_j)$	$\chi(A_m)$	$\chi(B_i, A_j)$	$\chi(B_k, A_m)$	$\mathbf{u} + \mathbf{v}$
p th	1	1	0	0	1	1	2
coordinate	0	0	1	1	-1	-1	-2

Case 2. $\mathbf{u} = \chi(B_i, A_j)$, $\mathbf{v} = \chi(B_i, A_m)$ or $\mathbf{u} = \chi(B_j, A_i)$, $\mathbf{v} = \chi(B_m, A_i)$, where $j \neq m$. In this case choose $k \in [4] \setminus \{i\}$. Then the pairs (B_i, A_j) and (B_k, A_m) are not compatible. As above, the vector $\chi(B_i, A_j) + \chi(B_k, A_m)$ has 2 or -2 at some position p . Suppose $p \in B_i \cap B_k$ and $p \notin A_j \cup A_m$. Then B_i has a 1 in p th position and each of the vectors $\chi(B_i, A_j)$ and $\chi(B_i, A_m)$ has a 1 in p th position as well. Therefore, $\mathbf{u} \oplus \mathbf{v}$ is undefined because $\mathbf{u} + \mathbf{v}$ has 2 in p th position. Similarly, in the case when $p \in A_j \cap A_m$ and $p \notin B_i \cup B_k$ the p th coordinate of $\mathbf{u} + \mathbf{v}$ is -2 . The case when $\mathbf{u} = \chi(B_j, A_i)$ and $\mathbf{v} = \chi(B_m, A_i)$ is similar.

Case 3. $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_k, B_m)$ or $\mathbf{u} = \chi(A_i, A_j)$, $\mathbf{v} = \chi(A_k, A_m)$, where $\{i, j, k, m\} = [4]$. By construction of M there exists $p \in [18]$ such that $p \in B_i \cap B_k$ and $p \notin B_j \cup B_m$ or $p \notin B_i \cup B_k$ and $p \in B_j \cap B_m$. So there is $p \in [18]$, such that $\mathbf{u} + \mathbf{v}$ has 2 or -2 in p th position. Thus $\mathbf{u} \oplus \mathbf{v}$ is undefined.

Case 4. $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_k, B_m)$ or $\mathbf{u} = \chi(A_i, A_j)$, $\mathbf{v} = \chi(A_k, A_m)$, where $i = k$ or $j = m$. If $i = k$ and $j = m$, then $\mathbf{u} \oplus \mathbf{v}$ is undefined. Consider the case $i = k$, $j \neq m$ and $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_i, B_m)$. Let $s = [4] \setminus \{i, j, m\}$. By construction of M either we have $p \in [18]$ such that $p \in B_i \cap B_s$ and $p \notin B_j \cup B_m$ or $p \notin B_i \cup B_s$ and $p \in B_j \cap B_m$. In both cases $\mathbf{u} + \mathbf{v}$ has 2 or -2 in position p .

Case 5. $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_k, B_m)$ or $\mathbf{u} = \chi(A_i, A_j)$, $\mathbf{v} = \chi(A_k, A_m)$, where $j = k$ or $i = m$. Suppose $j = k$. Since $i > j$ and $j > m$ we have $i > m$. This implies that $\chi(B_i, B_m)$ belongs to X . On the other hand $\mathbf{u} + \mathbf{v} = \chi(B_i) - \chi(B_m) = \chi(B_i, B_m)$. Therefore $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} \in X$. \square

Corollary 2. X' is closed under restricted sum.

Proof. We will have to consider the same five cases as in the Proposition 3. As above in the first four cases the restricted sum of vectors will be undefined. In the fifth case, when $\mathbf{u} = \chi(B'_i, B'_j)$, $\mathbf{v} = \chi(B'_k, B'_m)$ or $\mathbf{u} = \chi(A'_i, A'_j)$, $\mathbf{v} = \chi(A'_k, A'_m)$, where $j = k$ or $i = m$, we will have $\mathbf{u} + \mathbf{v} = \chi(B'_i) - \chi(B'_m) = \chi(B'_i, B'_m) \in X'$ or $\mathbf{u} + \mathbf{v} = \chi(A'_i) - \chi(A'_m) = \chi(A'_i, A'_m) \in X'$. \square

To satisfy conditions of Theorem 6 we need also to show that the intersection of the discrete cone $C(\preceq')$ and the hyperplane $H_{\mathbf{w}'}$ equals to $X' \cup -X'$. More explicitly we need to prove the following:

Proposition 4. Suppose $C, D \subseteq [26]$ are tied in \preceq' , that is $C \preceq' D$ and $D \preceq' C$. Then $\chi(C, D) \in X' \cup -X'$.

Proof. Assume to the contrary that there are two sets $C, D \in 2^{[26]}$ that have equal weights with respect to the corresponding system of weights defining \preceq' but $\chi(C, D) \notin X' \cup -X'$. The sets C and D have to contain some of the elements from $[26] \setminus [18]$ since w_1, \dots, w_{18} are linearly independent. Thus $C = C_1 \cup C_2$ and $D = D_1 \cup D_2$, where $C_1, D_1 \subseteq [18]$ and $C_2, D_2 \subseteq [26] \setminus [18]$ with C_2 and D_2 being nonempty. We have

$$0 = \chi(C, D) \cdot \mathbf{w}' = \chi(C_1, D_1) \cdot \mathbf{w} + \chi(C_2, D_2) \cdot \mathbf{w}^+,$$

where $\mathbf{w}^+ = (w_{19}, \dots, w_{26})^T$. By (17), we can express weights w_{19}, \dots, w_{26} as linear combinations with integer coefficients of N, K and w_1, \dots, w_{18} obtaining

$$\chi(C_2, D_2) \cdot \mathbf{w}^+ = \left(\sum_{i=1}^4 \gamma_i \chi(A_i) + \sum_{i=1}^4 \gamma_{4+i} \chi(B_i) \right) \cdot \mathbf{w} + \beta_1 N + \beta_2 K,$$

where $\gamma_i, \beta_j \in \mathbb{Z}$.

Clearly the expression in the bracket on the right-hand-side is just a vector with integer entries. Let us denote it α . Then

$$(19) \quad \chi(C_2, D_2) \cdot \mathbf{w}^+ = \alpha \cdot \mathbf{w} + \beta_1 N + \beta_2 K,$$

where $\alpha \in \mathbb{Z}^{18}$. We can now write $\chi(C, D) \cdot \mathbf{w}'$ in terms of \mathbf{w}, K and N :

$$0 = \chi(C, D) \cdot \mathbf{w}' = (\chi(C_1, D_1) + \alpha) \cdot \mathbf{w} + \beta_1 N + \beta_2 K.$$

We recap that K was chosen to be much greater than $\sum_{i \in [18]} w_i$ and N is much greater than K . So if β_1, β_2 are different from zero then $|\beta_1 N + \beta_2 K|$ is a very big number, which cannot be canceled out by $(\chi(C_1, D_1) + \alpha) \cdot \mathbf{w}$. Weights w_1, \dots, w_{18} are linearly independent, so for arbitrary $\mathbf{b} \in \mathbb{Z}^{18}$ the dot product $\mathbf{b} \cdot \mathbf{w}$ can be zero if and only if $\mathbf{b} = \mathbf{0}$. Hence

$$w(C) = w(D) \text{ iff } \chi(C_1, D_1) = -\alpha \text{ and } \beta_1 = 0, \beta_2 = 0.$$

Taking into account that $\chi(C_1, D_1)$ is a vector from T^{18} , we get

$$(20) \quad \alpha \notin T^{18} \implies w(C) \neq w(D).$$

We need the following two claims to finish the proof, their proofs are delegated to the next section.

Claim 1. *Suppose $\chi(C_1, D_1)$ belongs to $X \cup -X$. Then $\chi(C, D)$ belongs to $X' \cup -X'$.*

Claim 2. *If $\alpha \in T^{18}$, then α belongs to $X \cup -X$.*

Now let us show how with the help of these two claims the proof of Proposition 4 can be completed. The sets C and D have the same weight and this can happen only if α is a vector in T^{18} . By Claim 2 $\alpha \in X \cup -X$. The characteristic vector $\chi(C_1, D_1)$ is equal to $-\alpha$, hence $\chi(C_1, D_1) \in X \cup -X$. By Claim 1 we get $\chi(C, D) \in X' \cup -X'$, a contradiction. \square

Theorem 7. *There exists a linear qualitative probability order \sqsubseteq on $[26]$ and $T \subset [26]$ such that the initial segment $\Delta(\sqsubseteq, T)$ is not a threshold complex.*

Proof. By Corollary 2 and Proposition 4 all conditions of Theorem 6 are satisfied. Therefore $C(\preceq') \setminus (-X')$ is a discrete cone $C(\sqsubseteq)$, where \sqsubseteq is a almost representable linear qualitative probability order. By construction $A'_1 \sqsubset A'_2 \sqsubset A'_3 \sqsubset A'_4 \sqsubset B'_1 \sqsubset B'_2 \sqsubset B'_3 \sqsubset B'_4$ and thus $\Delta(\sqsubseteq, B'_1)$ is an initial segment, which is not a threshold complex. \square

Note that we have a significant degree of freedom in constructing such an example. The matrix M can be chosen in 2^{18} possible ways and we have not specified the linear qualitative probability order \preceq .

6. PROOFS OF CLAIM 1 AND CLAIM 2

Lets fix some notation first. Suppose $\mathbf{b} \in \mathbb{Z}^k$ and $\mathbf{x}_i \in \mathbb{Z}^n$ for $i \in [k]$. Then we define the product

$$\mathbf{b} \cdot (\mathbf{x}_1, \dots, \mathbf{x}_k) = \sum_{i \in [k]} b_i \mathbf{x}_i.$$

It resembles the dot product (the difference is that the second argument is a sequence of vectors) and is denoted in the same way. For a sequence of vectors $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ we also define $(\mathbf{x}_1, \dots, \mathbf{x}_k)_p = (\mathbf{x}_1^{(p)}, \dots, \mathbf{x}_k^{(p)})$, where $\mathbf{x}_i^{(j)}$ is the j th coordinate of vector \mathbf{x}_i .

We start with the following lemma.

Lemma 7. *Let $\mathbf{b} \in \mathbb{Z}^6$. Then*

$$\mathbf{b} \cdot (\chi(B_1, A_4), \chi(B_2, A_1), \chi(B_3, A_2), \chi(A_2, A_1), \chi(A_3, A_1), \chi(A_4, A_1)) = \mathbf{0}$$

if and only if $\mathbf{b} = \mathbf{0}$.

Proof. We know that the pairs (B_1, A_4) and (B_2, A_1) are not compatible. So there exists an element p that lies in the intersection $B_1 \cap B_2$ (or $A_1 \cap A_4$), but $p \notin A_4 \cup A_1$ ($p \notin B_1 \cup B_2$, respectively). We have exactly two copies of every element among A_1, \dots, A_4 and B_1, \dots, B_4 . Thus, the element p belongs to $A_2 \cap A_3$ ($B_3 \cap B_4$) and doesn't belong to $B_3 \cup B_4$ ($A_2 \cup A_3$). The following table illustrates this:

	$\chi(A_1)$	$\chi(A_2)$	$\chi(A_3)$	$\chi(A_4)$	$\chi(B_1)$	$\chi(B_2)$	$\chi(B_3)$	$\chi(B_4)$
p th	0	1	1	0	1	1	0	0
coordinate	1	0	0	1	0	0	1	1

Then at p th position we have

$$(\chi(B_1, A_4), \chi(B_2, A_1), \chi(B_3, A_2), \chi(A_2, A_1), \chi(A_3, A_1), \chi(A_4, A_1))_p = \pm(1, 1, -1, 1, 1, 0)$$

and hence

$$b_1 + b_2 - b_3 + b_4 + b_5 = 0.$$

From the fact that other pairs are not compatible we can get more equations relating b_1, \dots, b_6 :

$$\begin{aligned} b_1 - b_2 + b_3 - b_4 - b_6 &= 0 && \text{from } (B_1, A_4), (B_3, A_2); \\ -b_1 + b_2 + b_3 + b_5 + b_6 &= 0 && \text{from } (B_1, A_4), (B_4, A_3); \\ b_2 + b_5 + b_6 &= 0 && \text{from } (B_1, A_1), (B_2, A_2); \\ b_4 + b_6 &= 0 && \text{from } (B_1, A_1), (B_3, A_3); \\ b_3 + b_5 + b_6 &= 0 && \text{from } (B_1, A_1), (B_3, A_2). \end{aligned}$$

The obtained system of linear equations has only the zero solution. \square

Lemma 8. *Let $\mathbf{a} = (a_1, \dots, a_8)$ be a vector in \mathbb{Z}^8 whose every coordinate a_i has absolute value which is at most 100. Then $\mathbf{a} \cdot \mathbf{w}^+ = 0$ if and only if $\mathbf{a} = \mathbf{0}$.*

Proof. We first rewrite (17) in more convenient form:

$$\begin{aligned}
w_{19} &= N - K - (-\chi(B_1, A_4) + \chi(A_1)) \cdot \mathbf{w} \\
w_{20} &= N - K - (-\chi(B_1, A_4) - \chi(B_2, A_1) + \chi(A_2)) \cdot \mathbf{w} \\
w_{21} &= N - K - (-\chi(B_1, A_4) - \chi(B_2, A_1) - \chi(B_3, A_2) + \chi(A_3)) \cdot \mathbf{w} \\
w_{22} &= N - K - \chi(A_4) \cdot \mathbf{w} \\
(21) \quad w_{23} &= K - \chi(B_1, A_4) \cdot \mathbf{w} \\
w_{24} &= K - (\chi(B_1, A_4) + \chi(B_2, A_1)) \cdot \mathbf{w} \\
w_{25} &= K - (\chi(B_1, A_4) + \chi(B_2, A_1) + \chi(B_3, A_2)) \cdot \mathbf{w} \\
w_{26} &= K
\end{aligned}$$

We calculate the dot product $\mathbf{a} \cdot \mathbf{w}^+$ substituting the values of w_{19}, \dots, w_{26} from (21):

$$\begin{aligned}
(22) \quad 0 = \mathbf{a} \cdot \mathbf{w}^+ &= N \sum_{i \in [4]} a_i - K \left(\sum_{i \in [4]} a_i - \sum_{i \in [4]} a_{4+i} \right) \\
&\quad - \left[\chi(B_1, A_4) \left(\sum_{i=5}^7 a_i - \sum_{i=1}^3 a_i \right) + \chi(B_2, A_1) \left(\sum_{i=6}^7 a_i - \sum_{i=2}^3 a_i \right) \right. \\
&\quad \left. + \chi(B_3, A_2)(-a_3 + a_7) + \sum_{i \in [4]} a_i \chi(A_i) \right] \cdot \mathbf{w}.
\end{aligned}$$

The numbers N and K are very big and $\sum_{i \in [18]} w_i$ is small. Also $|a_i| \leq 100$. Hence the three summands cannot cancel each other. Therefore $\sum_{i \in [4]} a_i = 0$ and $\sum_{i \in [4]} a_{4+i} = 0$. The expression in the square brackets should be zero because the coordinates of \mathbf{w} are linearly independent.

We know that $a_1 = -a_2 - a_3 - a_4$, so the expression in the square brackets in (22) can be rewritten in the following form:

$$\begin{aligned}
(23) \quad &b_1 \chi(B_1, A_4) + b_2 \chi(B_2, A_1) + b_3 \chi(B_3, A_2) + \\
&a_2 \chi(A_2, A_1) + a_3 \chi(A_3, A_1) + a_4 \chi(A_4, A_1),
\end{aligned}$$

where $b_1 = \sum_{i=5}^7 a_i - \sum_{i=1}^3 a_i$, $b_2 = \sum_{i=6}^7 a_i - \sum_{i=2}^3 a_i$ and $b_3 = a_7 - a_3$.

By Lemma 7 we can see that expression (23) is zero iff $b_1 = 0$, $b_2 = 0$, $b_3 = 0$ and $a_2 = 0$, $a_3 = 0$, $a_4 = 0$ and this happens iff $\mathbf{a} = \mathbf{0}$. \square

Proof of Claim 1. Assume to the contrary that $\chi(C_1, D_1) \in X \cup -X$ and $\chi(C, D)$ does not belong to $X' \cup -X'$. Consider $\chi(\check{C}_1, \check{D}_1) \in X' \cup -X'$. We know that the weight of C is the same as the weight of D , and also that the weight of \check{C}_1 is the same as the weight of \check{D}_1 . This can be written as

$$\begin{aligned}
\chi(C_1, D_1) \cdot \mathbf{w} + \chi(C_2, D_2) \cdot \mathbf{w}^+ &= 0, \\
\chi(C_1, D_1) \cdot \mathbf{w} + \chi(\check{C}_1 \setminus C_1, \check{D}_1 \setminus D_1) \cdot \mathbf{w}^+ &= 0.
\end{aligned}$$

We can now see that

$$(\chi(\check{C}_1 \setminus C_1, \check{D}_1 \setminus D_1) - \chi(C_2, D_2)) \cdot \mathbf{w}^+ = 0.$$

The left-hand-side of the last equation is a linear combination of weights w_{19}, \dots, w_{26} . Due to Lemma 8 we conclude from here that

$$\chi(\check{C}_1 \setminus C_1, \check{D}_1 \setminus D_1) - \chi(C_2, D_2) = \mathbf{0}.$$

But this is equivalent to $\chi(C, D) = \chi(\check{C}_1, \check{D}_1) \in X$, which is a contradiction. \square

Proof of Claim 2. We remind the reader that α was defined in (19). Sets C and D has the same weight and we established that $\beta_1 = \beta_2 = 0$. So

$$\chi(C_2, D_2) \cdot \mathbf{w}^+ = \alpha \cdot \mathbf{w}.$$

If we look at the representation of the last eight weights in (21), we note that the weights $w_{19}, w_{20}, w_{21}, w_{22}$ are much heavier than the weights $w_{23}, w_{24}, w_{25}, w_{26}$. Hence $w(C) = w(D)$ implies

$$(24) \quad \begin{aligned} |C_2 \cap \{19, 20, 21, 22\}| &= |D_2 \cap \{19, 20, 21, 22\}| \text{ and} \\ |C_2 \cap \{23, 24, 25, 26\}| &= |D_2 \cap \{23, 24, 25, 26\}|. \end{aligned}$$

That is C and D have equal number of super-heavy weights and equal number of heavy ones.

Without loss of generality we can assume that $C_2 \cap D_2$ is empty. Similar to derivation in the proof of Lemma 8, the vector α can be expressed as

$$(25) \quad \alpha = a_1\chi(B_1, A_4) + a_2\chi(B_2, A_1) + a_3\chi(B_3, A_2) + \sum_{i \in [4]} b_i\chi(A_i)$$

for some $a_i, b_j \in \mathbb{Z}$. The characteristic vectors $\chi(A_1), \dots, \chi(A_4)$ participate in the representations of super-heavy elements w_{19}, \dots, w_{22} only. Hence $b_i = 1$ iff element $18 + i \in C_2$ and $b_i = -1$ iff element $18 + i \in D_2$. Without loss of generality we can assume that $C_2 \cap D_2 = \emptyset$. By (24) we can see that if C_2 contains some super-heavy element $p \in \{19, \dots, 22\}$ with $\chi(A_k)$, $k \in [4]$, in the representation of w_p , then D_2 has a super-heavy $q \in \{19, \dots, 22\}$, $q \neq p$ with $\chi(A_t)$, $t \in [4] \setminus \{k\}$ in representation of w_q . In such case $b_k = -b_t = 1$ and

$$b_k\chi(A_k) + b_t\chi(A_t) = \chi(A_k, A_t).$$

By (24) the number of super-heavy element in C_2 is the same as the number of super-heavy elements in D_2 . Therefore (25) can be rewritten in the following way:

$$(26) \quad \alpha = a_1\chi(B_1, A_4) + a_2\chi(B_2, A_1) + a_3\chi(B_3, A_2) + a_4\chi(A_i, A_p) + a_5\chi(A_k, A_t),$$

where $a_1, a_2, a_3 \in \mathbb{Z}$; $a_4, a_5 \in \{0, 1\}$ and $\{i, k, t, p\} = [4]$.

Now the series of technical facts will finish the proof.

Fact 1. Suppose $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ and $|\{i, k, t\}| = |\{j, m, s\}| = 3$. Then

$$a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) \in T^{18}$$

if and only if

$$(27) \quad \mathbf{a} \in \{(0, 0, 0), (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1), (1, 1, 1), (-1, -1, -1)\}.$$

Proof. The pairs $((B_j, A_i), (B_m, A_k))$, $((B_j, A_i), (B_s, A_t))$ and $((B_m, A_k), (B_s, A_t))$ are not compatible. Using the same technique as in the proofs of Proposition 3 and Lemma 7 and watching a particular coordinate we get

$$(a_1 + a_2 - a_3), (a_1 - a_2 + a_3), (-a_1 + a_2 + a_3) \in T,$$

respectively. The absolute value of the sum of every two of these terms is at most two. Add the first term to the third. Then $|2a_2| \leq 2$ or, equivalently, $|a_2| \leq 1$. In a similar way we can show that $|a_3| \leq 1$ and $|a_1| \leq 1$. The only vectors that satisfy all the conditions above are those listed in (27). \square

Fact 2. Suppose $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ and $|\{i, k, t\}| = |\{j, m, s\}| = 3$. Then

$$a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) + \chi(A_k, A_t) \in T^{18}$$

if and only if

$$(28) \quad \mathbf{a} \in \{(0, 0, 0), (0, 1, 0), (0, 0, -1), (0, 1, -1)\}.$$

Proof. Considering non-compatible pairs $((B_m, A_k), (B_s, A_t)), ((B_j, A_i), (B_m, A_k)), ((B_j, A_i), (B_s, A_t)), ((B_j, A_k), (B_s, A_i)), ((B_j, A_t), (B_m, A_i))$, we get the inclusions

$$(-a_1 + a_2 + a_3), (a_1 + a_2 - a_3 - 1), (a_1 - a_2 + a_3 + 1), (a_1 - 1), (a_1 + 1) \in T,$$

respectively. We can see that $|2a_2 - 1| \leq 2$ and $|2a_3 + 1| \leq 2$ and $a_1 = 0$. So a_2 can be only 0 or 1 and a_3 can have values -1 or 0 . \square

Fact 3. Suppose $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ and $\{i, k, t, p\} = [4]$ and $|\{j, m, s\}| = 3$. Then

$$a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) + \chi(A_i, A_p) \in T^{18}$$

if and only if

$$a \in \{(0, 0, 0), (1, 0, 0), (1, 1, 1), (2, 1, 1)\}.$$

Proof. Let $\ell \in [4] \setminus \{j, m, s\}$. From consideration of the following non-compatible pairs

$$((B_j, A_i), (B_m, A_k)), ((B_j, A_i), (B_s, A_t)), ((B_m, A_k), (B_s, A_t)), ((B_j, A_i), (B_m, A_t)), ((B_j, A_i), (B_m, A_p)), ((B_j, A_i), (B_s, A_p)), ((B_s, A_t), (B_\ell, A_i))$$

we get the following inclusions

$$(a_1 + a_2 - a_3 - 1), (a_1 - a_2 + a_3 - 1), (-a_1 + a_2 + a_3), (a_1 - 1), (a_1 - a_3), (a_1 - a_2), (a_2 - a_3 + 1) \in T,$$

respectively. So we have $|2a_3 - 1| \leq 2$ (from the second and the third inclusions) and $|2a_2 - 1| \leq 2$ (from the first and the third inclusions) from which we immediately get $a_2, a_3 \in \{1, 0\}$. We also get $a_1 \in \{2, 1, 0\}$ (by the forth inclusion).

- If $a_1 = 2$, then by the fifth and sixth inclusions $a_3 = 1$ and $a_2 = 1$.
- If $a_1 = 1$, then a_2 can be either zero or one. If $a_2 = 0$ then we have $\chi(B_j, A_i) + a_3\chi(B_s, A_t) + \chi(A_i, A_p) = \chi(B_j, A_p) + a_3\chi(B_s, A_t)$. By Fact 1, a_3 can be zero only. On the other hand, if $a_2 = 1$, then $a_3 = 1$ by the seventh inclusion.
- If $a_1 = 0$ then a_2 can be a 0 or a 1. Suppose $a_2 = 0$. Then $a_3 = 0$ by the first two inclusions. Assume $a_2 = 1$. Then $a_3 = 0$ by the third inclusion and on the other hand $a_3 = 1$ by the second inclusion, a contradiction.

This proves the statement. \square

Fact 4. Suppose $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ and $\{i, k, t, p\} = [4]$ and $|\{j, m, s\}| = 3$. Then

$$a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) + \chi(A_i, A_p) + \chi(A_k, A_t) \notin T^{18}.$$

Proof. Let $\ell \in [4] \setminus \{j, m, s\}$. Using the same technique as above from consideration of non-compatible pairs

$$\begin{aligned} &((B_j, A_i), (B_m, A_t)), ((B_s, A_t), (B_j, A_k)), ((B_j, A_i), (B_s, A_t)), \\ &((B_m, A_k), (B_s, A_t)), ((B_j, A_i), (B_m, A_p)), ((B_j, A_i), (B_\ell, A_k)) \end{aligned}$$

we obtain inclusions:

$$a_1, a_3, (a_1 - a_2 + a_3), (-a_1 + a_2 + a_3), (a_1 - a_3), (a_1 - a_3 - 2) \in T,$$

respectively.

From the last two inclusions we can see that $a_1 - a_3 = 1$. This, together with the first and the second inclusions, imply $(a_1, a_3) \in \{(1, 0), (0, -1)\}$. Suppose $(a_1, a_3) = (1, 0)$. Then

$$\chi(B_j, A_i) + a_2\chi(B_m, A_k) + \chi(A_i, A_p) + \chi(A_k, A_t) = \chi(B_j, A_p) + a_2\chi(B_m, A_k) + \chi(A_k, A_t).$$

By Fact 3, it doesn't belong to T^{18} for any value of a_2 .

Suppose now that $(a_1, a_3) = (0, -1)$. Then by the third and the forth inclusions a_2 can be only zero. Then $\mathbf{a} = (0, 0, -1)$ and

$$-\chi(B_s, A_t) + \chi(A_i, A_p) + \chi(A_k, A_t) = -\chi(B_s, A_k) + \chi(A_i, A_p).$$

However, by Fact 3 the right-hand-side of this equation is not a vector of T^{18} . \square

Fact 5. Suppose $\mathbf{a} \in \mathbb{Z}^5$ and

$$\mathbf{v} = a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) + a_4\chi(A_i, A_p) + a_5\chi(A_k, A_t).$$

If $a_4, a_5 \in \{0, 1, -1\}$ and $v \in T^{18}$, then v belongs to X or $-X$.

Proof. First of all, we will find the possible values of \mathbf{a} in case $\mathbf{v} \in T^{18}$. By Facts 1–4 one can see that $\mathbf{v} \in T^{18}$ iff \mathbf{a} belongs to the set

$$\begin{aligned} Q = \{ &(0, 0, 0, 0, 0), (\pm 1, 0, 0, 0, 0), (0, \pm 1, 0, 0, 0), (0, 0, \pm 1, 0, 0), (1, 1, 1, 0, 0), \\ &(0, 0, 0, \pm 1, 0), (\pm 1, 0, 0, \pm 1, 0), (\pm 1, \pm 1, \pm 1, \pm 1, 0), (\pm 2, \pm 1, \pm 1, \pm 1, 0), \\ &(0, 0, 0, 0, \pm 1), (0, \pm 1, 0, 0, \pm 1), (0, 0, \mp 1, 0, \pm 1), (0, \pm 1, \mp 1, 0, \pm 1) \}. \end{aligned}$$

By the construction of \preceq the sequence $(A_1, \dots, A_4; B_1, \dots, B_4)$ is a trading transform. So for every $\{i_1, \dots, i_4\} = \{j_1, \dots, j_4\} = [4]$ the equation

$$(29) \quad \chi(B_{i_1}, A_{j_1}) + \chi(B_{i_2}, A_{j_2}) + \chi(B_{i_3}, A_{j_3}) + \chi(B_{i_4}, A_{j_4}) = 0.$$

holds. Taking (29) into account one can show, that for every $\mathbf{a} \in Q$, vector \mathbf{v} belongs to X or $-X$. For example, if $\mathbf{a} = (2, 1, 1, 1, 0)$ then

$$\begin{aligned} 2\chi(B_j, A_i) + \chi(B_m, A_k) + \chi(B_s, A_t) + \chi(A_i, A_p) = \\ \chi(B_j, A_i) - \chi(B_\ell, A_p) + \chi(A_i, A_p) = \chi(B_j, B_\ell), \end{aligned}$$

where $\ell \in [4] \setminus \{j, m, s\}$. \square

One can see that \mathbf{v} from the Fact 5 is the general form of α . Hence $\alpha \in T^{18}$ if and only if $\alpha \in X \cup -X$ which is Claim 2. \square

7. ACYCLIC GAMES AND A CONJECTURED CHARACTERIZATION

So far we have shown that the initial segment complexes strictly contain the threshold complexes and are strictly contained within the shifted complexes. In this section we introduce some ideas from the theory of simple games to formulate a conjecture that characterizes initial segment complexes. The idea in this section is to start with a simplicial complex and see if there is a natural linear order available on $2^{[n]}$ which gives a qualitative probability order and has the original complex as an initial segment. We will follow the presentation of Taylor and Zwicker Taylor & Zwicker (1999).

Let $\Delta \subseteq 2^{[n]}$ be a simplicial complex. Define the *Winder desirability relation*, \leq_W , on $2^{[n]}$ by $A \leq_W B$ if and only if for every $Z \subseteq [n] \setminus ((A \setminus B) \cup (B \setminus A))$ we have that

$$(A \setminus B) \cup Z \notin \Delta \Rightarrow (B \setminus A) \cup Z \notin \Delta.$$

Furthermore define the *Winder existential ordering*, \prec_W , on $2^{[n]}$ to be

$$A \prec_W B \iff \text{It is not the case that } B \leq_W A.$$

Definition 5. A simplicial complex Δ is called *strongly acyclic* if there are no k -cycles

$$A_1 \prec_W A_2 \prec_W \cdots \prec_W A_k \prec_W A_1$$

for any k in the Winder existential ordering.

Theorem 8. Suppose \preceq is a qualitative probability order on $2^{[n]}$ and $T \in 2^{[n]}$. Then the initial segment $\Delta(\preceq, T)$ is strongly acyclic.

Proof. Let $\Delta = \Delta(\preceq, T)$. It follows from the definition that $A \prec_W B$ if and only if there exists a $Z \in [n] \setminus ((A \setminus B) \cup (B \setminus A))$ such that $(A \setminus B) \cup Z \in \Delta$ and $(B \setminus A) \cup Z \notin \Delta$. From the definition of Δ it follows that

$$(A \setminus B) \cup Z \prec (B \setminus A) \cup Z$$

which, by de Finetti's axiom 2, implies

$$A \setminus B \prec B \setminus A$$

and hence, again by de Finetti's axiom 2,

$$A \prec B.$$

Thus a k -cycle

$$A_1 \prec_W \cdots \prec_W A_k \prec_W A_1$$

in Δ would imply a k -cycle

$$A_1 \prec \cdots \prec A_k \prec A_1$$

which contradicts that \prec is a total order. \square

Conjecture 1. A simplicial complex Δ is an initial segment complex if and only if it is strongly acyclic.

We will return momentarily to give some support for Conjecture 1. First, however, it is worth noting that the necessary condition of being strongly acyclic from Theorem 8 allows us to see that there is little relationship between being an initial segment complex and satisfying the conditions CC_k^* .

Corollary 3. *For every $M > 0$ there exist simplicial complexes that satisfy CC_M^* but are not initial segment complexes.*

Proof. Taylor and Zwicker Taylor & Zwicker (1999) construct a family of complexes $\{G_k\}$, which they call Gabelman games that satisfy CC_{k-1}^* but not CC_k^* . They then show (Taylor & Zwicker, 1999, Corollary 4.10.7) that none of these examples are strongly acyclic. The result then follows from Theorem 8. \square

Our evidence in support of Conjecture 1 is based on the idea that the Winder existential order can be used to produce the related qualitative probability order for strongly acyclic complexes. Here are two lemmas that give some support for this belief:

Lemma 9. *If Δ is a simplicial complex with $A \in \Delta$ and $B \notin \Delta$ then $A \prec_W B$.*

Proof. Let $Z = A \cap B$. Then

$$\begin{aligned} (A \setminus B) \cup Z &= A \in \Delta \\ (B \setminus A) \cup Z &= B \notin \Delta, \end{aligned}$$

and so $A \prec_W B$. \square

Lemma 10. *For any Δ , the Winder existential order \prec_W satisfies the property*

$$A \prec_W B \iff A \cup D \prec_W B \cup D$$

for all D disjoint from $A \cup B$.

Proof. See (Taylor & Zwicker, 1999, Proposition 4.7.8). \square

This pair of lemmas leads to a slightly stronger version of Conjecture 1.

Conjecture 2. *If Δ is strongly acyclic then there exists an extension of \prec_W to a qualitative probability order.*

What are the barriers to proving Conjecture 2? The Winder order need not be transitive. In fact there are examples of threshold complexes for which \prec_W is not transitive (Taylor & Zwicker, 1999, Proposition 4.7.3). Thus one would have to work with the transitive closure of \prec_W , which does not seem to have a tractable description. In particular we do not know if the analogue to Lemma 10 holds for the transitive closure of \prec_W .

8. CONCLUSION

In this paper we have begun the study of a class of simplicial complexes that are combinatorial generalizations of threshold complexes derived from qualitative probability orders. We have shown that this new class of complexes strictly contains the threshold complexes and is strictly contained in the shifted complexes. Although we can not give a complete characterization of the complexes in question, we conjecture that they are the strongly acyclic complexes that arise in the study of cooperative games. We hope that this conjecture will draw attention to the ideas developed in game theory which we believe to be too often neglected in the combinatorial literature.

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